Thus, the system with commuting damping and stiffness matrices is similar to a diagonal system and possesses normal modes as predicted by theorem 2.

Discussion

The concepts of classical normal modes for a general discrete linear system were defined. The well-known theorem developed by Rayleigh for existence of normal modes in symmetric damped systems was extended to a more general class of dynamic systems, i.e., asymmetric systems with simple coefficient matrices. Some results developed by Caughey and O'Kelly on classical normal modes in symmetric and asymmetric systems were discussed and used to generate results on classical normal modes, similar to those available for symmetric second-order systems.

It should be noted that the results developed here are not necessarily more general than those developed by Caughey and O'Kelly. However, the new conditions for classical normal modes in asymmetric systems, presented here, can be checked in a systematic way and, hence, may be more computationally attractive.

The discretized model of a damped, flexible, rotatory shaft, and a hypothetical three-degree-of-freedom asymmetric system were used to illustrate the developed results.

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References

[1] Lord Rayleigh, *The Theory of Sound*, Vol. 1, Dover Press, New York, 1945, p. 130.

[2] Caughey, T. K. and O'Kelly, M. F. J., "Classical Normal Modes in Damped Linear Dynamic Systems," *ASME Journal of Applied Mechanics*, Vol. 32, June 1965, pp. 583-588.

[3] O'Kelly, M. E. J., "Vibration of Viscously Damped Linear Dynamic Systems," Ph.D. Dissertation, California Institute of Technology, May 1964, pp. 47-50.

[4] Caughey, T. K. and O'Kelly, M. E. J., "General Theory of Vibration of Damped Linear Systems," Dynamics Laboratory Rept., California Institute of Technology, June 1963, pp. 14-41.

[5] Lancaster, P., *Eigenvalues and Eigenvectors, Theory of Matrices*, 1st ed., Academic Press, Inc., New York, 1969, pp. 81,82.

[6] Tausskey, O., "Positive Definite Matrices and Their Role in the Study of the Characteristic Roots of General Matrices," *Advances in Mathematics*, Vol. 2, 1968, pp. 175-186.

[7] Inman, D. J., "Dynamics of Asymmetric Non-Conservative Systems," *Journal of Applied Mechanics*, Vol. 150, March 1983, pp. 199-203.

[8] Shieh, R. C. and Masur, E. F., "Some General Principle of Dynamic Instability of Solid Bodies," *Zeitschrift fuer Angewandte Mathematik und Physik*, Vol. 19, Sept. 1968, pp. 927-941.

Thermal Effect on Axisymmetric Vibrations of an Orthotropic Circular Plate of Variable Thickness

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Introduction

MUCH work has been done on the vibrations of orthotropic circular plates,[1-3] but none of it has considered the thermal effect on such vibrations. It is well known[4] that, in the presence of a constant thermal gradient, the elastic coefficients of homogeneous materials become functions of the space variables. Fanconneau and Marangoni[5] have investigated the effect of the nonhomogeneity caused by a thermal gradient on the natural frequencies of simply supported plates of uniform thickness. Recently, Tomar and Tewari[6] have studied the effect of the thermal gradient on the frequencies of a circular plate of linearly varying thickness.

Thermally induced vibrations of elastic plates are of great interest in aircraft and machine designs and also in chemical, nuclear, and astronautical engineering. Nonisotropic plates of nonuniform thickness are being widely used in the design of modern missiles, space vehicles, aircraft wings, and numerous composite engineering machines. The analysis presented here studies the effect of a constant thermal gradient on frequencies of an orthotropic circular plate of variable thickness. The deflection function is expressed here in the form of an infinite series from which frequencies corresponding to the first three vibration modes are obtained for various values of the thickness variation, taper constant, and temperature gradient.

Analysis and Equation of Motion

It is assumed that the circular plate of orthotropic materials is subjected to a steady temperature in the radial direction

$$T = T_0(1-R) \tag{1}$$

where T denotes the temperature excess above the reference temperature at any point at a distance $R=r/a$ from the center of the circular plate of radius a and T_0 denotes the temperature excess at $r=0$ above the reference temperature at any point on the circumference of the circular plate, i.e., at $r=a$ or $R=1$.

The temperature dependence of Young's moduli in the r and Θ directions for most of the orthotropic material is given by

$$E_r(T) = E_1(1-\gamma T), \qquad E_\Theta(T) = E_2(1-\gamma T) \tag{2}$$

where E_1 and E_2 are the values of Young's moduli, respectively along the r and Θ directions at the reference temperature, i.e., at $T=0$.

Taking the reference temperature as that at the edge of the circular plate, i.e, at $R=1$, Young's moduli in view of Eqs. (1) and (2) becomes

$$E_r(R) = E_1[1-\alpha(1-R)], \qquad E_\Theta(R) = E_2[1-\alpha(1-R)] \tag{3}$$

where $\alpha=\gamma T_0$ $(0\le\alpha\le 1)$, which is a parameter known as the temperature gradient.

The governing differential equation of axisymmetric motion of an orthotropic circular plate of variable thickness is[2]

$$\begin{aligned} &D_r w_{,rrrr} + 2[(D_r + rD_{r,r})/r]w_{,rrr} \\ &\quad + [(-D_\Theta + r(2+\nu_\Theta)D_{r,r} + r^2 D_{r,rr})/r^2]w_{,rr} \\ &\quad + [(D_\Theta - rD_{\Theta,r} + r^2\nu_\Theta D_{r,rr})/r^3]w_{,r} + \rho h w_{,tt} = 0 \end{aligned} \tag{4}$$

where $D_r = E_r h^3/12(1-\nu_r\nu_\Theta)$ and $D_\Theta = E_\Theta h^3/12(1-\nu_r\nu_\Theta)$. Also, ν_r and ν_Θ are the Poisson's ratio in the r and Θ directions, respectively, w the transverse deflection, ρ the mass density per unit volume, t the time, and h the thickness. A comma followed by a subscript denotes partial differentiation with respect to that variable.

Since the axis of the plate coincides with the radial direction, one may find that thickness h, D_r, and D_Θ of the plate become functions of r alone. For free transverse vibrations of

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the plate, w can be expressed as

$$w=\bar{W}(r)e^{ipt} \tag{5}$$

where p is radian frequency.

Introducing the nondimensional variables

$$H=\frac{h}{a},\ W=\frac{\bar{W}}{a},\ R=\frac{r}{a},\ D_R=\frac{D_r}{a^3},\ \text{and}\ D_\Theta=\frac{D_\Theta}{a^3} \tag{6}$$

one gets expression for rigidities as

$$D_R=\frac{E_{11}H^3}{12}[1-\alpha(1-R)],\ D_\Theta=\frac{E_{22}H^3}{12}[1-\alpha(1-R)] \tag{7}$$

where $E_{11}=E_1/(1-\nu_r\nu_\Theta)$ and $E_{22}=E_2/(1-\nu_r\nu_\Theta)$.

Assuming the thickness variation of the plate as

$$H(R)=H_0(1-\beta R^n) \tag{8}$$

where β is the taper constant and $H_0=H|_{R=0}$. The equation of motion is then

$$\begin{aligned}
&[1-\alpha(1-R)](1-\beta R^n)^2W_{,RRRR}+2\{[[1-\alpha(1-R)]\\
&\quad\times(1-\beta R^n)^2-3n\beta R^n[1-\alpha(1-R)](1-\beta R^n)\\
&\quad+\alpha R(1-\beta R^n)^2]/R\}W_{,RRR}+\{[-\ell^2[1-\alpha(1-R)]\\
&\quad\times(1-\beta R^n)^2-3(2+m^2)n\beta R^n[1-\alpha(1-R)](1-\beta R^n)\\
&\quad+(2+m^2)\alpha R(1-\beta R^n)^2+6n^2\beta^2R^{2n}[1-\alpha(1-R)]\\
&\quad-3n(n-1)\beta R^n[1-\alpha(1-R)](1-\beta R^n)-6n\alpha\beta R^{n+1}\\
&\quad\times(1-\beta R^n)]/R^2\}W_{,RR}+\{[\ell^2[1-\alpha(1-R)](1-\beta R^n)^2\\
&\quad+3n\ell^2\beta R^n[1-\alpha(1-R)](1-\beta R^n)-\ell^2\alpha R(1-\beta R^n)^2\\
&\quad+6n^2m^2\beta^2R^{2n}[1-\alpha(1-R)]-3n(n-1)m^2\beta R^n\\
&\quad\times[1-\alpha(1-R)](1-\beta R^n)-6nm^2\alpha\beta R^{n+1}\\
&\quad\times(1-\beta R^n)]/R^3\}W_{,R}=\lambda^2W
\end{aligned} \tag{9}$$

where the frequency parameter is

$$\lambda^2=\frac{p^2a^2}{E_{11}/\rho}\cdot\frac{12}{H_0^2}\ \text{and}\ \ell^2=\frac{E_{22}}{E_{11}},\qquad m^2=\nu_\Theta \tag{10}$$

Solution and Frequency Equation

A series solution for W is assumed in the form

$$W=\sum_{k=0}^{\infty}a_kR^{C+k}\qquad a_0\neq 0 \tag{11}$$

where C is the exponent of singularity.

For Eq. (11) to be the solution, the coefficients of the various powers of R in the expression obtained by substituting Eq. (11) into Eq. (9) must be identically zero. Thus, by equating to zero the coefficient of the lowest power of R, the following indicial roots are obtained: $C=0$, 2, $1+\ell$, and $1-\ell$.

It may be seen that the series corresponding to $C=0$ will also contain the series corresponding to $C=2$. Also in the series corresponding to $C=1-\ell$, $\ell>1$ vanishes because of its singularity at $R=0$; and in the series for $C=1-\ell$, $\ell<1$ will be contained in the series corresponding to $C=1+\ell$. Thus, the complete solution takes the form

$$W=\sum_{k=0}^{\infty}b_kR^k+\sum_{k=0}^{\infty}d_kR^{k+1+\ell} \tag{12}$$

where b_k and d_k correspond to a_k evaluated for $C=0$ and $1+\ell$, respectively. The constants b_k and d_k $(k\geq 1)$ are determined from the recurrence relation,

$$\begin{aligned}
&(C+k)(C+k-2)[(C+k-1)^2-\ell^2](1-\alpha)a_k\\
&\quad+(C+k-1)\{(C+k-2)[(C+k-3)(C+k)+2+m^2]\\
&\quad-\ell^2\}\alpha a_{k-1}+U(k-n)(C+k-n)\{(C+k-1-n)\\
&\quad\times[-2(C+k-2-n)(C+k+5-n)+2\ell^2-18-6m^2]\\
&\quad+4\ell^2-6m^2\}\beta(1-\alpha)a_{k-n}+U[(k-1)-n]\\
&\quad\times(C+k-1-n)\{(C+k-2-n)[-2(C+k-3-n)\\
&\quad\times(C+k+6-n)+2\ell^2-34-8m^2]+6\ell^2-18m^2\}\alpha\beta a_{k-1-n}\\
&\quad+U(k-2n)(C+k-2n)\{(C+k-1-2n)\\
&\quad\times[(C+k-2-2n)(C+k+11-2n)-\ell^2+42+6m^2]\\
&\quad-5\ell^2+30m^2\}\beta^2(1-\alpha)a_{k-2n}+U[(k-1)-2n]\\
&\quad\times(C+k+2n)\{(C+k-2-2n)[(C+k-3-2n)\\
&\quad\times(C+k+12-2n)-\ell^2+56+7m^2]-6\ell^2\\
&\quad+42m^2\}\alpha\beta^2a_{k-1-2n}-U(k-4)\lambda^2a_{k-4}=0,\quad k\geq 1
\end{aligned} \tag{13}$$

where the unit step function

$$\begin{aligned}
U(k_1-n_1)&=1\ \text{for}\ k_1-n_1\geq 0\\
&=0\ \text{for}\ k_1-n_1<0
\end{aligned}$$

in which $k_1=k$ or $k-1$ and $n_1=n$, $2n$, or 4.

Thus solution of Eq. (12) becomes

$$W=b_0F(R,\lambda^2)+d_0G(R,\lambda^2) \tag{14}$$

where

$$F(R,\lambda^2)=1+\sum_{k=3}^{\infty}b_kR^k,\quad G(R,\lambda^2)=1+\sum_{k=1}^{\infty}d_kR^{k+1+\ell} \tag{15}$$

Application of the technique used by Lamb[7] shows that solution of Eq. (14) is convergent for $|\beta|<1$ and $\alpha<0.5$.

At $r=a$, i.e., $R=1$, the boundary conditions are for a clamped plate,

$$W|_{R=1}=\frac{\partial W}{\partial R}\bigg|_{R=1}=0 \tag{16a}$$

and for a simply supported plate,

$$W|_{R=1}=M_R|_{R=1}=0$$

i.e.,

$$W|_{R=1}=\left(\frac{\partial^2W}{\partial R^2}+\frac{1}{R}\nu_\Theta\frac{\partial W}{\partial R}\right)\bigg|_{R=1}=0 \tag{16b}$$

Applying the boundary conditions (16) to Eq. (14), one gets the frequency equations as,

$$\begin{vmatrix}F(1,\lambda^2) & G(1,\lambda^2)\\ F'(1,\lambda^2) & G'(1,\lambda^2)\end{vmatrix}=0\quad\text{(clamped plate)} \tag{17}$$

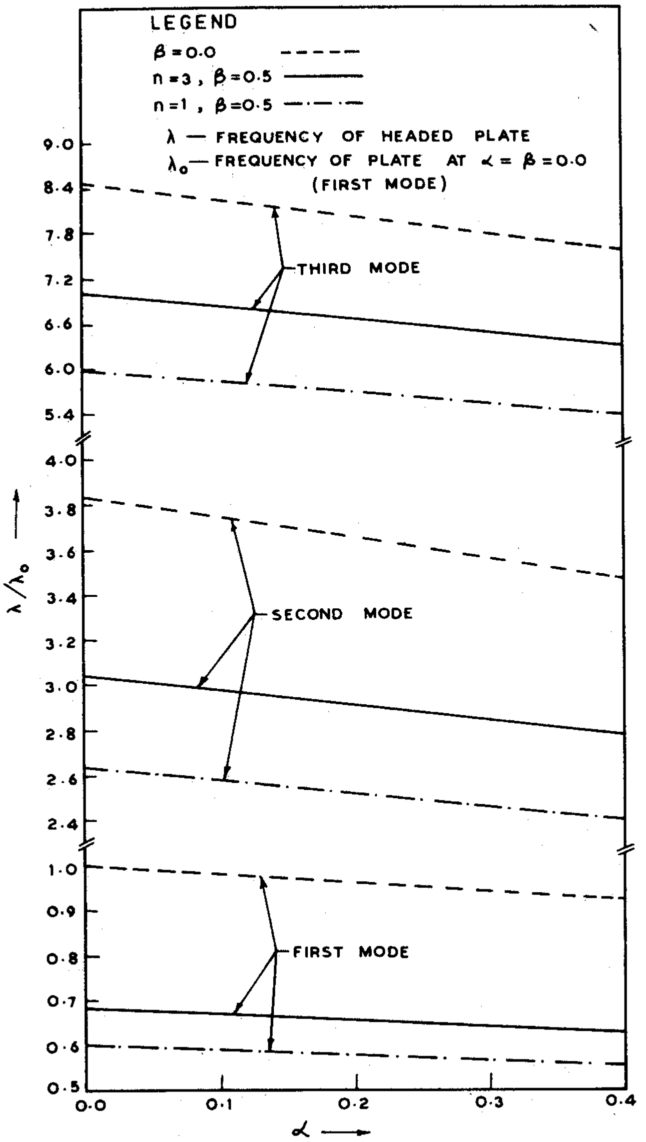

Fig. 1 Variation of frequency with temperature gradient for clamped orthotropic circular plate.

$$\begin{vmatrix} F(1,\lambda^2) & G(1,\lambda^2) \\ F''(1,\lambda^2)+\nu_\theta F'(1,\lambda^2) & G''(1,\lambda^2)+\nu_\theta G'(1,\lambda^2) \end{vmatrix} = 0$$

(simply supported plate) (18)

Here, the prime denotes the partial differentiation with respect to R.

Results and Discussion

The frequency equations (17) and (18) are transcendental in λ^2 from which an infinite number of roots can be determined. Since the results for clamped and simply supported boundary conditions are only slightly different quantitatively but have the same overall qualitative trends, we have given the results for the clamped case only. The frequencies λ corresponding to the first three modes of vibration of a clamped orthotropic circular plate have been computed for various values of thickness variation n, temperature gradient α, and taper constant β. The orthotropic material parameters are taken[2] as $\ell^2 = 1.44$ and $m^2 = 0.3$. It is concluded that frequencies in the first three modes of vibration decrease with the increase in the temperature gradient. The frequencies corresponding to the first three modes of vibration for various values of n, α, and β are plotted in Fig. 1. It is observed that the frequencies decrease with the increasing values of taper constant β. By taking $\alpha = 0.0$, our results reduced to the results of Ref. 2. The shape of transverse deflection curves for the first three modes of vibration for three combinations of n, α, and β is plotted in Fig. 2.

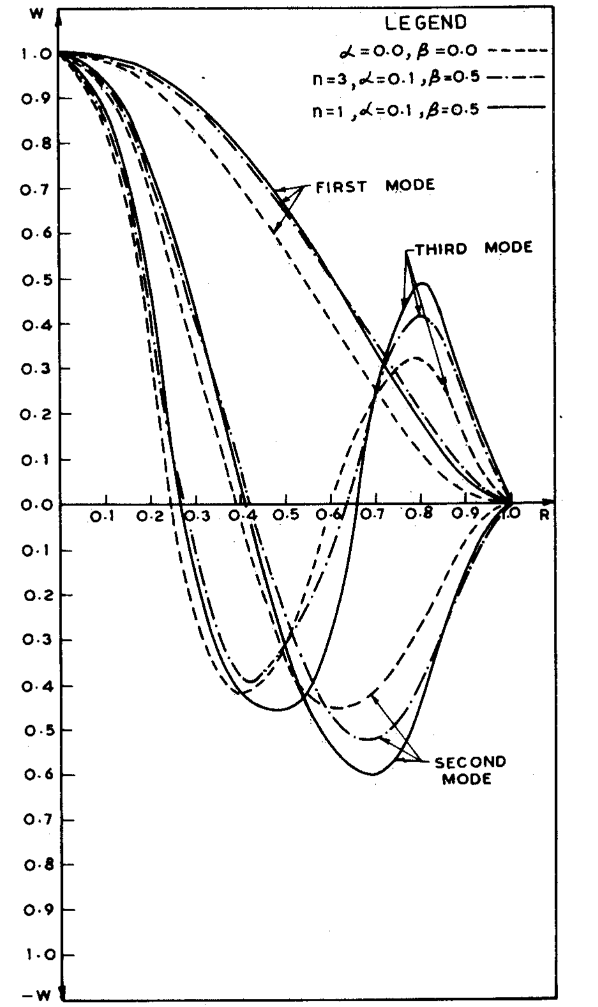

Fig. 2 Transverse deflection of clamped orthotropic circular plate.

References

[1] Pandalai, K. A. V. and Patel, S. A., "Natural Frequencies of Orthotropic Circular Plates," *AIAA Journal*, Vol. 3, April 1965, pp. 780-781.

[2] Salzman, A. P. and Patel, S. A., "Natural Frequencies of Orthotropic Circular Plate of Variable Thickness," *AIAA Journal*, Vol. 6, Aug. 1968, pp. 1625-1626.

[3] Leissa, A. W., "Vibration of Plates," NASA SP 160, 1969.

[4] Hoff, N. J., *High Temperature Effect in Aircraft Structures*, Pergamon Press, New York, 1958, p. 62.

[5] Fanconneau, G. and Marangoni, R. D., "Effect of a Thermal Gradient on the Natural Frequencies of a Rectangular Plate," *International Journal of Mechanical Sciences*, Vol. 12, 1970, pp. 113-122.

[6] Tomar, J. S. and Tewari, V. S., "Effect of Thermal Gradient on Frequencies of a Circular Plate of Linearly Varying Thickness," *Journal of Nonequilibrium Thermodynamics*, Vol. 6, 1981, pp. 115-122.

[7] Lamb, H., *Hydrodynamics*, 1945, Dover Publications, New York, 1945, p. 335.